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Towards an exhaustive classification of the star–triangle relation: I

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Abstract. We study the star–triangle relation for two-component spin models on a square lattice, in order to classify exhaustively such a relation. For this purpose, we obtain necessary conditions for the star–triangle relation to be satisfied, by considering the commutation of transfer matrices of arbitrary size N, for small N. The number of such conditions is compared with the number of relevant parameters of the model, and we are led to distinguish two very different cases. Only the first one is dealt with in this paper.

1. Introduction

The star–triangle relation (STR) has been shown to be a crucial element in the study of exactly solved two-dimensional models, in statistical mechanics as well as in field theory (Baxter 1980a, 1982; e.g. Bazhanov and Stroganov 1981). The underlying reason is that the (local) STR is a sufficient condition for the commutation of (global) transfer matrices; the study of models solved by this commutation property is therefore reduced to the study of an *a priori* simple local relation. Two remarks are here in order. First of all, the number of such models is, at the present time, relatively small. Secondly, the '*a priori* simple' local relation leads in fact to a set of trilinear homogeneous equations, which is quite complicated (Maillard 1983). Moreover, this set seems to over-determine the few solutions that can exist, as can be seen from particular examples. It therefore appears natural to try to classify exhaustively the non-trivial solutions of the STR. Two large families of models are concerned in this classification: vertex models and spin models.

For the 16-vertex model, the classification of the STR has been carried out by Krichever (1981) using a sufficient condition for the STR to be satisfied; this author used the fact that a pure tensor product is transformed into another pure tensor product by the local Boltzmann weight associated to the vertex (two-body S matrix in S matrix theory) (Krichever 1981, Jaekel and Maillard 1983). Krichever's results can be summarised as follows: the non-trivial models that satisfy the STR are mainly (a) the symmetric 8-vertex model (Baxter 1972, 1973), (b) the asymmetric 6-vertex model (McCoy and Wu 1968), (c) the Fan–Wu free fermion model (Fan and Wu 1970, Felderhof 1972, 1973a, b). The symmetric 6-vertex and Ising models are subcases of the above models, and correspond to zero field conditions.

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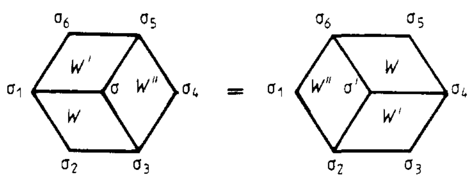
|| Note that this model is referred to as Felderhof's model in Krichever's (1971) paper.

As for two-component spin models, such a classification does not exist on the market. It is the ultimate goal of this series of two papers. The vertex—and spin—star–triangle relations do not in general coincide, as exemplified by the hard hexagon model (Baxter 1980b). Krichever's analysis cannot be straightforwardly generalised to spin models since, in this case, there is no obvious† pure tensor product stability property. We therefore choose a different strategy. We will find necessary conditions for the STR to be satisfied, by considering the commutation of transfer matrices of small size N. We will mainly use the stability of the STR under the inversion relation (Stroganov 1979).

The plan of the paper is as follows: § 2 briefly recalls the STR and its consequences for the transfer matrices of arbitrary size N. In § 3, we extract from the small N cases necessary conditions for the STR to be satisfied. The number of such conditions is compared, in the framework of some gauge invariance, with the number of relevant parameters of the model (§ 4). This study will lead us to distinguish two cases: the first one corresponds to trivial solutions of the STR and is studied in this paper. The non-trivial solutions will be studied in the following paper, hereafter referred to as (II).

2. Star–triangle relation. Commutation of transfer matrices

Let us give a graphical illustration as well as a definition of the STR. We denote by W, W', W'' the Boltzmann weights associated to each elementary cell of a square lattice (it is understood that W, W', W'' correspond to three different choices of parameters of a given model). The STR means that the partition function of the two graphs below are equal:



All spin variables are Ising-like and $\{\sigma_i\}$, $i=1,2,\ldots,6$, are fixed spins and $\{\sigma,\sigma'\}$ are summed upon. Analytically this relation reads

$$\sum_{\sigma} W(\sigma_1,\sigma_2,\sigma_3,\sigma)\,W'(\sigma_6,\sigma_1,\sigma,\sigma_5)\,W''(\sigma,\sigma_3,\sigma_4,\sigma_5)$$
$$=\sum_{\sigma'} W(\sigma_6,\sigma',\sigma_4,\sigma_5)\,W'(\sigma',\sigma_2,\sigma_3,\sigma_4)\,W''(\sigma_1,\sigma_2,\sigma',\sigma_6). \qquad (1)$$

If the STR (1) holds, and provided one takes periodic boundary conditions, the transfer matrices associated with the Boltzmann weights W and W' commute, even for arbitrary size N (Baxter 1980a, Kasteleyn 1975). The transfer matrices‡ will be denoted respectively by $T_N(W)$ and $T_N(W')$. What is remarkable in all known soluble cases

† Such a property may in fact exist for spin models, as revealed by some cases solved on disorder lines (Baxter, private communication).

‡ The transfer matrix $T_N(W)$ transfers from one horizontal row to the next one. It may therefore be called a horizontal transfer matrix. The STR (1) implies a similar commutation property for the vertical transfer matrices, that we denote by $\tilde{T}_N(W)$ and $\tilde{T}_N(W')$.

is that the STR (1) or its consequence

$$(T_N(W), T_N(W')) = 0 \qquad \text{for all } N \tag{2}$$

implies (see e.g. Jaekel and Maillard 1983)

$$\varphi_{i,N}(W) = \varphi_{i,N}(W') \tag{3}$$

where $\varphi_{i,N}$ is an algebraic function of the parameters of the model (the index i indicates that there may be several functions for a given N). If one tries to solve equations (1) by eliminating the parameters associated with W'', one gets a vanishing determinant condition like

$$\det(\text{matrix}(W, W')) = 0$$

because of the linear homogeneous character of (1). This condition cannot *a priori* be factorised, and equation (3) must indeed be considered as remarkable. Moreover, (3) allows the model to be foliated: in the case of the Baxter model, one recovers the known characterisation of elliptic functions as intersecting quadrics (Baxter 1982). One should also mention that there are always trivial solutions of the STR (1), such as $W = \text{constant} \times W'$ (the transfer matrix commutes with itself). There are other uninteresting cases where the STR is satisfied for any weight W, W', W'': they often correspond to one- or zero-dimensional models in disguise.

3. Study of the star–triangle relation for the IRF model

3.1. Definition of the model

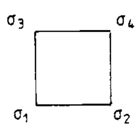
Let us recall the definition of the interaction-round-a-face (IRF) model (Baxter 1980a): we consider Ising spins $\{\sigma_i = \pm i\}$ at the corners of an elementary cell (square), see figure 1. The IRF model is the most general model corresponding to these $2^4 = 16$ spin configurations. We therefore have 16 parameters, denoted by $a, b, c, d, \ldots, m, n, o, p$ (figure 1). Throughout this paper, we assume that none of these parameters is zero (no excluded configurations).

3.2. Commutation of transfer matrices for small size N

We now study in detail the consequences of (2) for small N ($N = 1, 2, 3, 4, \ldots$).

(a) $N = 1$. Note that, due to the periodic boundary conditions, $T_1(W)$ is a 2×2 matrix (figure 2(a)): $T_1(W)$ transfers from spin σ_1 to spin σ_2 and reads

$$T_1(W) = \begin{array}{c|cc} \sigma_2 \backslash \sigma_1 & + & - \\ \hline + & a & d \\ - & m & p \end{array}$$



$$\longrightarrow \quad \begin{array}{c|cccc} (\sigma_3,\sigma_4) \backslash (\sigma_1,\sigma_2) & ++ & +- & -+ & -- \\ \hline ++ & a & b & c & d \\ +- & e & f & g & h \\ -+ & i & j & k & l \\ -- & m & n & o & p \end{array}$$

Figure 1. The IRF model.

Figure 2. Commutation of transfer matrices: (*a*) $N=1$, (*b*) $N=2$.

Equation (2) then yields

$$d/m = d'/m', \tag{4}$$

$$(a-p)/d = (a'-p')/d'. \tag{5}$$

(b) $N=2$. For $N=2$, $T_2(W)$ is a 4×4 matrix, which transfers from spins (σ_1, σ_2) to spins (σ_3, σ_4) (figure 2(*b*)), and equation (2) leads to

$$2BD'+CH' = 2B'D+CH, \tag{6}$$

$$DB'+GI' = D'B+G'I, \tag{7}$$

$$2GI'+CH' = 2G'I+C'H, \tag{8}$$

$$DX'+HG' = D'X+H'G, \tag{9}$$

$$BX'+CI' = B'X+C'I, \tag{10}$$

$$H(Y'-X')+2ID' = H'(Y-X)+2I'D, \tag{11}$$

$$C(Y'-X')+2BG' = C'(Y-X)+2B'G, \tag{12}$$

$$IY'+HB' = I'Y+H'B, \tag{13}$$

$$GY'+DC' = G'Y+D'C, \tag{14}$$

where the following notations have been used: $B=bc$, $C=d^2$, $D=ei$, $G=hl$, $H=m^2$, $I=no$, $X=fk+gj-a^2$, $Y=fk+gj-p^2$. Using (4), equations (6)–(8) yield

$$B/D = B'/D', \qquad I/G = I'/G'. \tag{15a, b}$$

Let us now consider equations (9), (10). We set $H=\lambda C$, $B=\mu D$, $I=\rho G$, and we get

$$DX'+\lambda CG' = D'X+\lambda C'G, \tag{16}$$

$$\mu DX'+\rho CG' = \mu D'X+\rho C'G. \tag{17}$$

This implies

$$(\rho-\lambda\mu)CG' = (\rho-\lambda\mu)C'G.$$

If $\rho-\lambda\mu\neq0$, we have

$$G/C = G'/C', \qquad D/X = D'/X'. \tag{18a,b}$$

The study of (11)–(14) leads to similar results. We may thus conclude that

$$\rho-\lambda\mu\neq0 \rightarrow \frac{B}{B'}=\frac{D}{D'}=\frac{X}{X'}=\frac{C}{C'}=\frac{I}{I'}=\frac{G}{G'}=\frac{Y}{Y'}. \tag{19}$$

On the contrary, if $\rho-\lambda\mu=0$, we still get equations (15) but the remaining equations (9), (11), (13) cannot be factorised.

(c) $N=3,4$. Using the same method as in Jaekel and Maillard (1983), we obtain from $(T_3(W), T_3(W'))=0$ four relations (two by two conjugate)

$$\varphi_i(W)=\varphi_i(W')$$

with

$$\varphi_1(W)=(gjd+\omega kbh+\omega^2 flc)/(gjm+\omega foi+\omega^2 nek), \tag{20}$$

$$\varphi_2(W)=\frac{kf(a-p)+\omega(big-hoj)+\omega^2(jec-gnl)}{gjd+\omega flc+\omega^2 kbh}, \tag{21}$$

where $\omega^3=1$. A little extra work would yield more relations; for instance, at $N=4$, we get, among many,

$$\varphi(W)=(fjco-bkng)/(ejkh-fgil)=\varphi(W'). \tag{22}$$

4. Gauge invariance and number of variables

Let us consider the (horizontal) transfer matrix $T_N(W)$. Due to the periodic boundary conditions (PBC), $T_N(W)$ is left unchanged by the transformation D_1:

$$W(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \xrightarrow{D_1} \frac{D_1(\sigma_1,\sigma_3)}{D_1(\sigma_2,\sigma_4)} W(\sigma_1,\sigma_2,\sigma_3,\sigma_4). \tag{23}$$

The positions of the $\{\sigma_i\}$ are shown in figure 1. The invariance of $T_N(W)$ under D_1 holds because the $D_1(\sigma_i,\sigma_j)$ factors cancel two by two, even the first and last ones (PBC). Since $T_N(W)$ is unchanged, so is the partition function Z under (23). Since $\sigma_i=\pm1$, the transformation D_1 is a three-parameter transformation. On the other hand, we could have defined another transformation D_2:

$$W(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \xrightarrow{D_2} \frac{D_2(\sigma_1,\sigma_2)}{D_2(\sigma_3,\sigma_4)} W(\sigma_1,\sigma_2,\sigma_3,\sigma_4). \tag{24}$$

D_2 leaves Z unchanged but modifies $T_N(W)$; it is also a three-parameter transformation. Note that D_1 and D_2 overlap and that the intersection $(D_1\cap D_2)$ of these two 'gauge' transformations is a one-parameter family. Given the IRF model, one may conclude that the use of D_1 and D_2 allows one to gauge away *at most* $3+3-1=5$ parameters. These transformations are similar to the weak graph duality for vertex models (Gaaf and Hijmans 1975).

Let us now compare, *in the case* $\rho\neq\lambda\mu$, the number of conditions obtained from (2), and the number of relevant variables among the 16 (homogeneous) initial ones, that is among 15 parameters. The gauge transformation D_1 which leaves $T_N(W)$ invariant also leaves (3) invariant. Therefore one may fully use D_1 and gauge away three irrelevant parameters, without modifying the number of conditions $\varphi_a(W)=\varphi_\alpha(W')$. Summing up the number of such conditions for $N=1,2,3,4$, we find at least 13 of them, whereas the number of remaining parameters is 12. If one assumes that all conditions $\varphi_\alpha(W)=\varphi_\alpha(W')$ are independent, the only possible solution of the STR (1) is $W=\text{constant}\times W'$ and that concludes our discussion of the case $\rho\neq\lambda\mu$. To get non-trivial solutions, we have to consider the case $\rho\neq\lambda\mu$ which is the subject of paper (II), *or* some possible solutions with $\rho\neq\lambda\mu$ with non-independent conditions $\varphi_\alpha(W)=\varphi_\alpha(W')$. We shall see in (II) that such solutions may only exist when the inversion relation does not exist for the model.

5. Conclusion

In this paper, we have initiated a possible strategy for studying the STR in the case of two-component spin models. This strategy has been shown to work in the case $\rho \neq \lambda\mu$, with the result that there does not exist a non-trivial solution of the STR, provided the conditions $\varphi_\alpha(W) = \varphi_\alpha(W')$ are independent. For the other case, $\rho = \lambda\mu$, this approach does not allow one to conclude. In paper (II), we will consider this case by using the inversion relation: due to its compatibility with the STR, the inversion relation will enable us to generate new invariants $\varphi_\alpha(W)$ from the old ones.

References

Baxter R J 1972 *Ann. Phys.* **70** 193
—— 1973 *Ann. Phys.* **71** 1
—— 1980a in *Fundamental problems in statistical mechanics* V (Amsterdam: North-Holland)
—— 1980b *J. Phys. A: Math. Gen.* **13** L61
—— 1982 *Exactly solved models in statistical mechanics* (New York: Academic)
Bazhanov V V and Stroganov Y G 1981 *Phys. Lett.* **105B** 278
Fan C and Wu F Y 1970 *Phys. Rev.* B **2** 723
Felderhof B U 1972 *Physica* **65** 421
—— 1973a *Physica* **66** 279
—— 1973b *Physica* **66** 509
Gaaf A and Hijmans J 1975 *Physica* **80A** 149
Jaekel M and Maillard J M 1983 *J. Phys. A: Math. Gen.* **16** 3105
Kasteleyn P W 1975 in *Fundamental problems in statistical mechanics* III (Amsterdam: North-Holland)
Krichever I M 1981 *Funct. Anal. Appl. Meth.* **15** 92
McCoy B and Wu T T 1968 *Nuovo Cimento* **56B** 311
Maillard J M 1983 *Note CEA N-2335, Relation triangle étoile en mécanique statistique*
Stroganov Y G 1979 *Phys. Lett.* **74A** 116